The Joy of Mathematics - 5

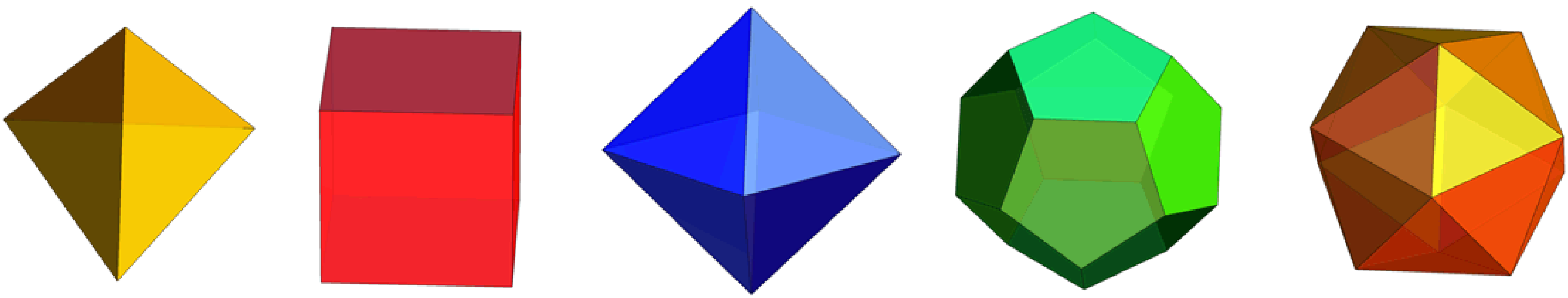
Shapes and solids and a touch of the "Origamis"

Stella's Polyhedra Bookcase
http://www.software3d.com/Bookcase.php

The Platonic Solids Song!

http://www.youtube.com/watch?v=C36h00d7xGs

In Geometry we looked at the regular polygons – only 5 can be used to make regular solid polyhedra with all sides the same:



Tetrahedron
4 Faces
Equilateral
Triangles

Hexahedron
(Cube)
6 Faces
Squares

Octahedron
8 Faces
Equilateral
Triangles

Dodecahedron
12 Faces
Regular
Pentagons

Icosahedron
20 Faces
Equilateral
Triangles

These are know as the Platonic solids after the Greek philosopher Plato

http://world.mathigon.org/Polygons_and_Polyhedra

Plato’s interpretation

Plato wrote about them in the dialogue Timaeus *c*.360 B.C.

- Earth was associated with the cube ,
- air with the octahedron,
- water with the icosahedron,
- and fire with the tetrahedron.

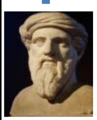
There was intuitive justification for these associations: the heat of fire feels sharp and stabbing (like little tetrahedra). Air is made of the octahedron; its minuscule components are so smooth that one can barely feel it. Water, the icosahedron, flows out of one's hand when picked up, as if it is made of tiny little balls. By contrast, a highly nonspherical solid, the hexahedron (cube) represents "earth". These clumsy little solids cause dirt to crumble and break when picked up in stark difference to the smooth flow of water. Moreover, the cube's being the only regular solid that tesselates Euclidean space was believed to cause the solidity of the Earth. The fifth Platonic solid, the dodecahedron, Plato obscurely remarks, "...the god used for arranging the constellations on the whole heaven".

Aristotle added a fifth element, aithêr (aether in Latin, "ether" in English) and postulated that the heavens were made of this element, but he had no interest in matching it with Plato's fifth solid.

http://en.wikipedia.org/wiki/Platonic_solid

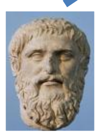
Greek mathematicians and polyhedra

600 BC | 500 BC | 400 BC | 300 BC | 200 BC



Pythagoras

- 570 - 495 BC
- Dodecahedron – requires his theorem to construct



Plato

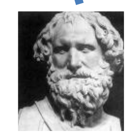
- 424/423 – 348/347 BC
- made the list of the five platonic polyhedra

Theaetetus

- .417 – 369 BC
- octahedron and icosahedron; first to construct five regular polyhedra

Euclid

- 300 BC
- studied the five regular polyhedra and wrote about them in “Elements.” He proved that no more than five can exist



Archimedes

- c. 287 - c. 212 BC
- made the list of the thirteen semi-regular polyhedra (Archimedean polyhedra)

See http://www.storyofmathematics.com/mathematicians.html for a comprehensive timeline of Mathematicians

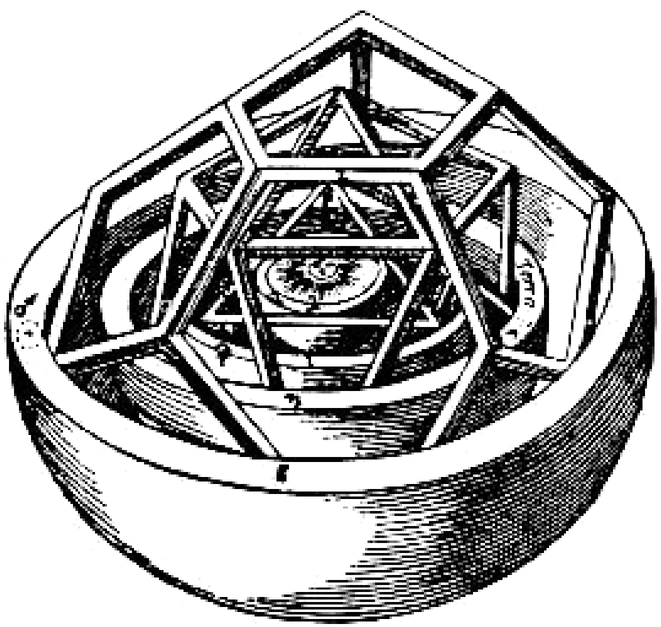
Other Important Discoverers

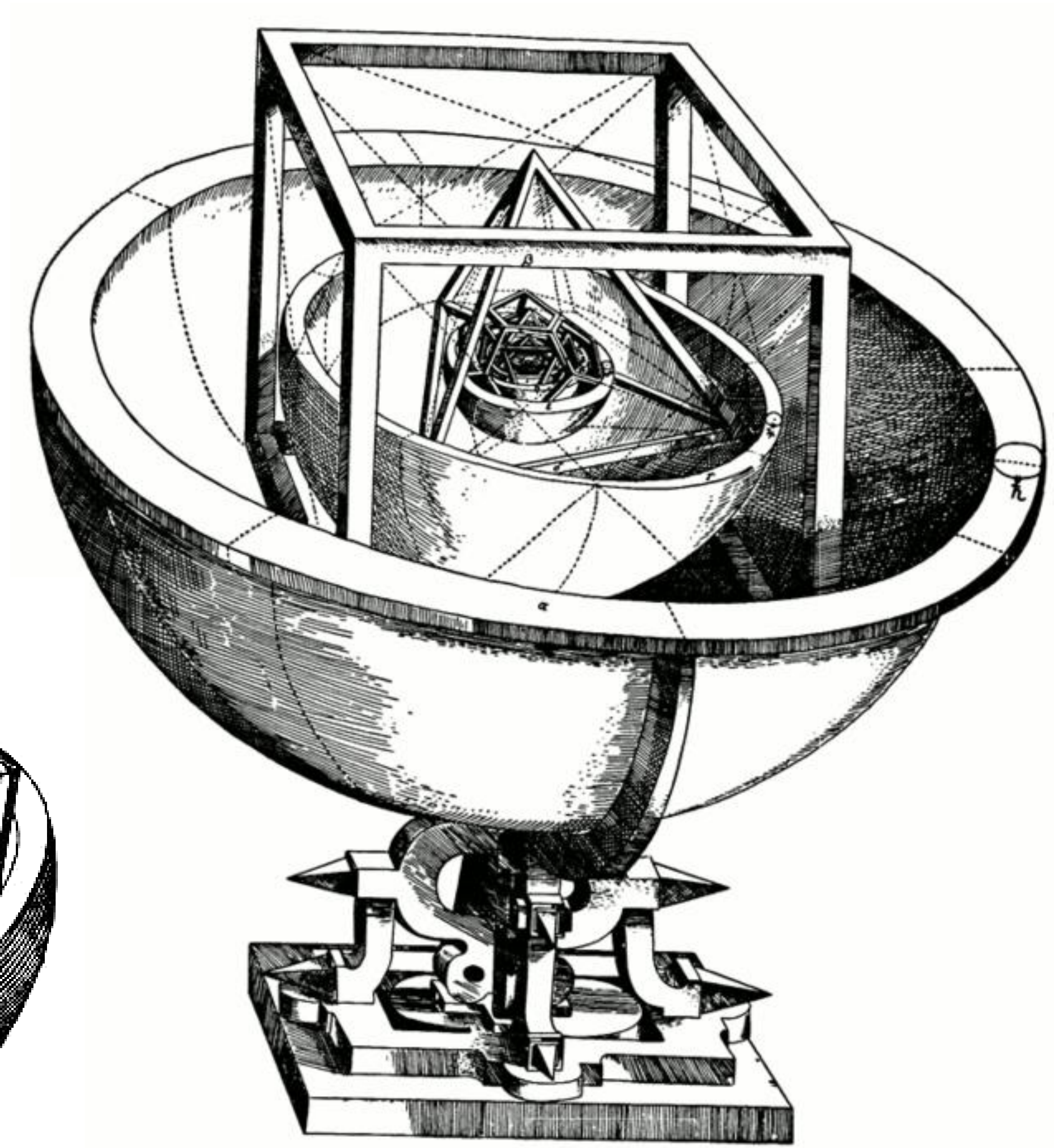
Johannes Kepler (27 December 27 1571 – 15 November 1630): systemized and extended what was known about polyhedra. He defined classes of polyhedra, discovered the members of the class, and proved that his set was complete.

Leonhard Euler (15 April 1707 – 18 September 1783): discovered Euler's Polyhedron Formula, which links the numbers of vertices, edges and faces for the convex polyhedra

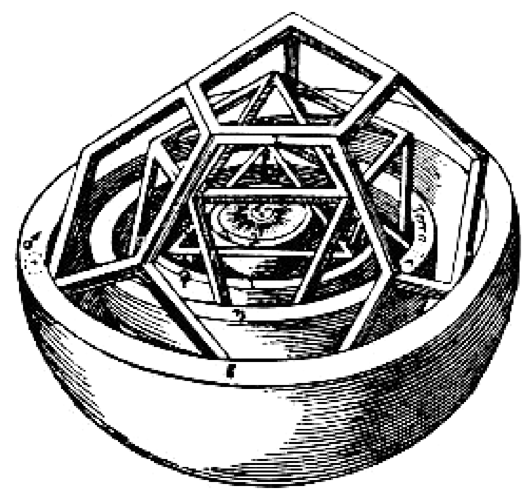
http://math2033.uark.edu/wiki/index.php/History_of_Polyhedra

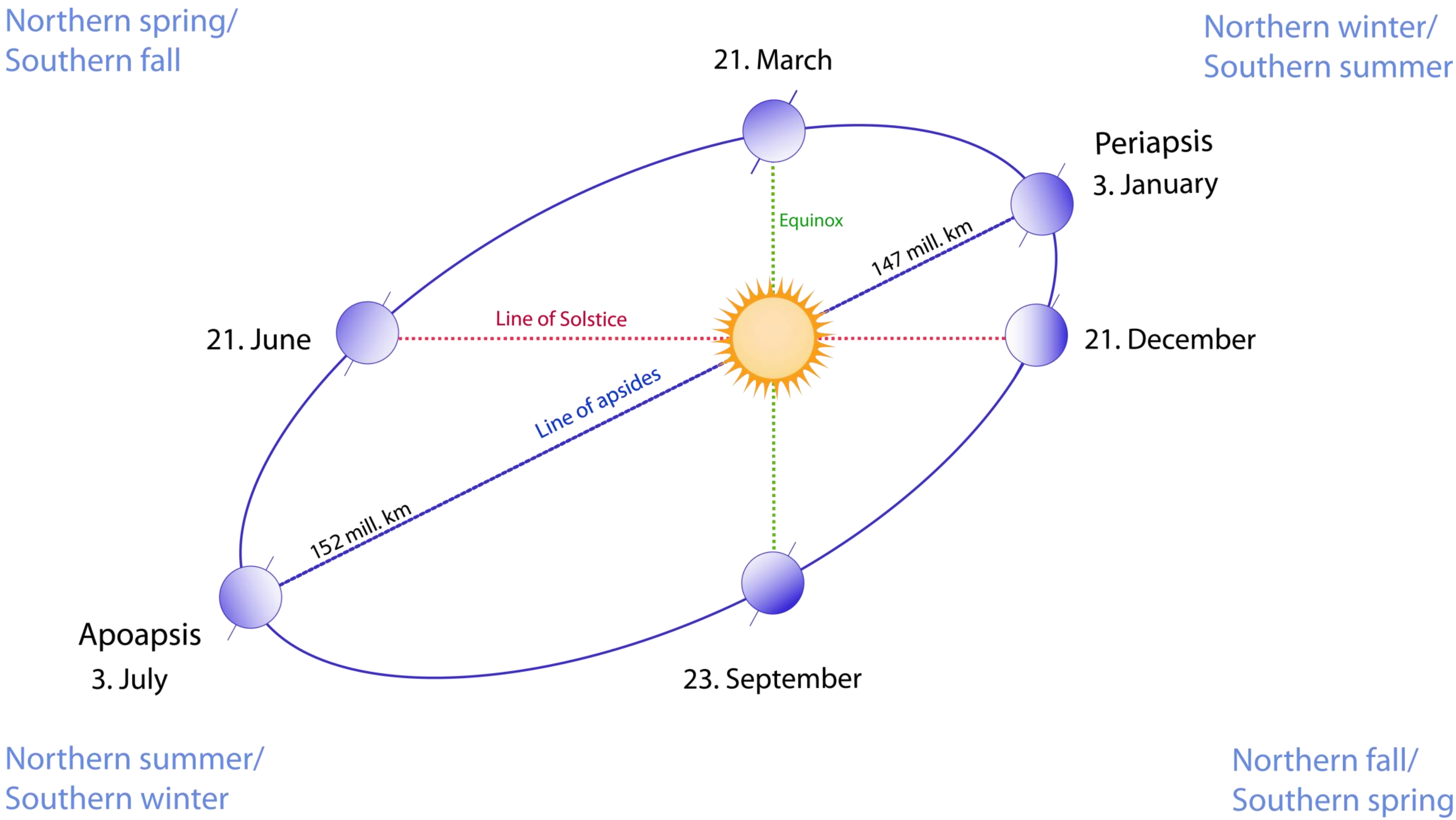
Kepler’s ***Mysterium Cosmographicum***





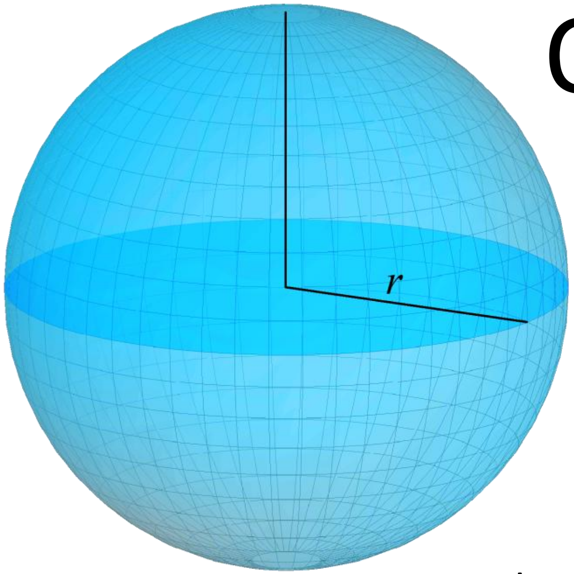
He found that each of the five Platonic solids could be uniquely inscribed and circumscribed by spherical orbs; nesting these solids, each encased in a sphere, within one another would produce six layers, corresponding to the six known planets—Mercury, Venus, Earth, Mars, Jupiter, and Saturn. By ordering the solids correctly —octahedron, icosahedron, dodecahedron, tetrahedron, cube—Kepler found that the spheres could be placed at intervals corresponding (within the accuracy limits of available astronomical observations) to the relative sizes of each planet's path, assuming the planets circle the Sun.



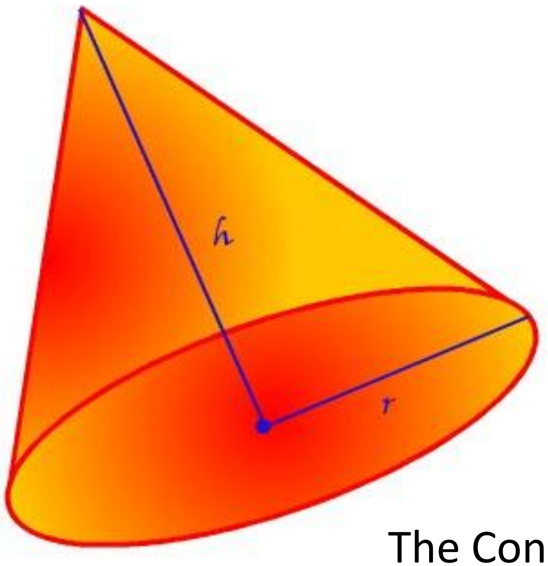


In 1609, Kepler published the first two of his three laws of planetary motion. The first law states:
"The orbit of every planet is an ellipse with the sun at a focus."

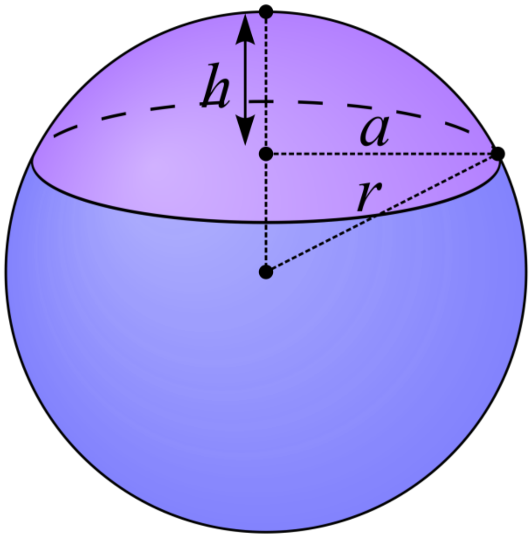
Other Solids



The Sphere

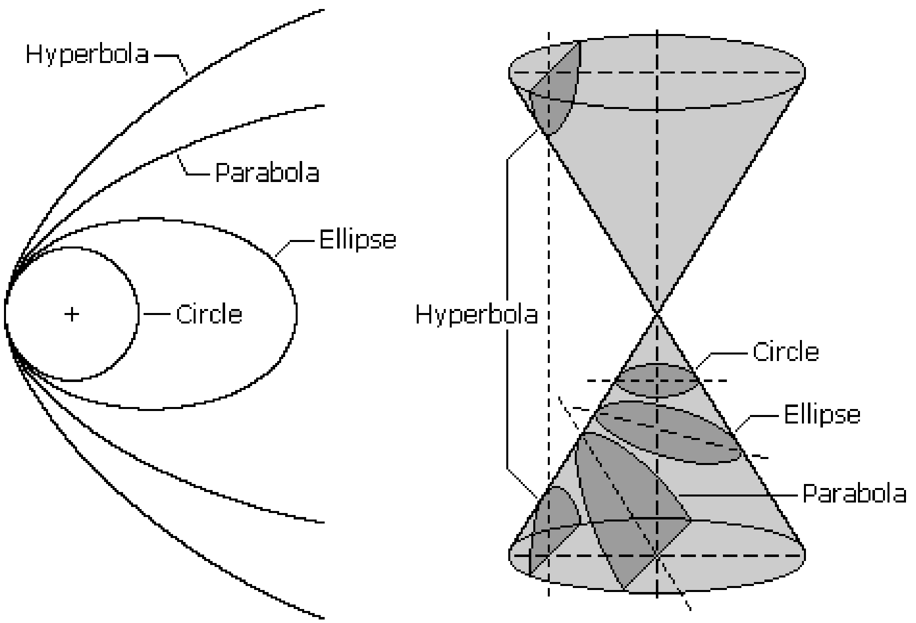


The Cone



Spherical sections are always circles

Conic sections



Euler Polyhedron Formula

$$V - E + F = 2$$

Number of Vertices minus the number of edges plus the number of faces always equals 2

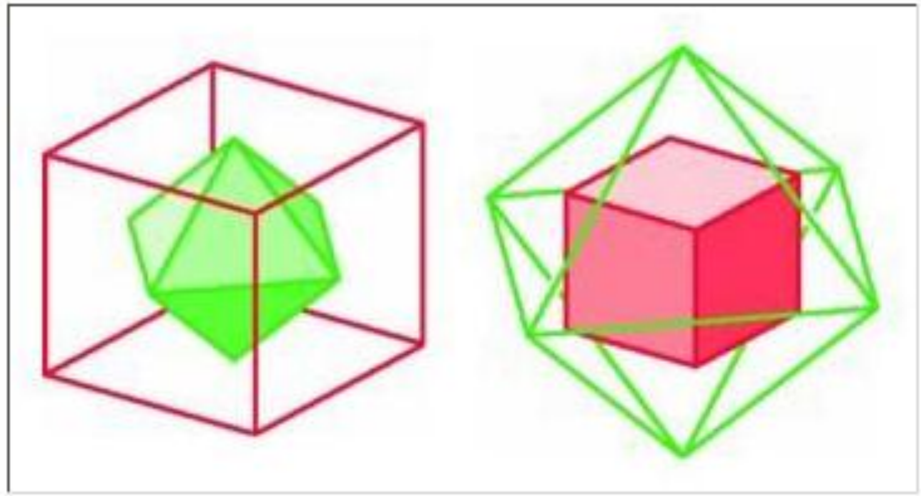
This is a version of a more general equation **F + V – E = χ** *Where* **χ** *is called the* "**Euler Characteristic**" *which can apply to other objects, e.g. The Moebius strip and the torus.*

http://www.mathsisfun.com/geometry/eulers-formula.html

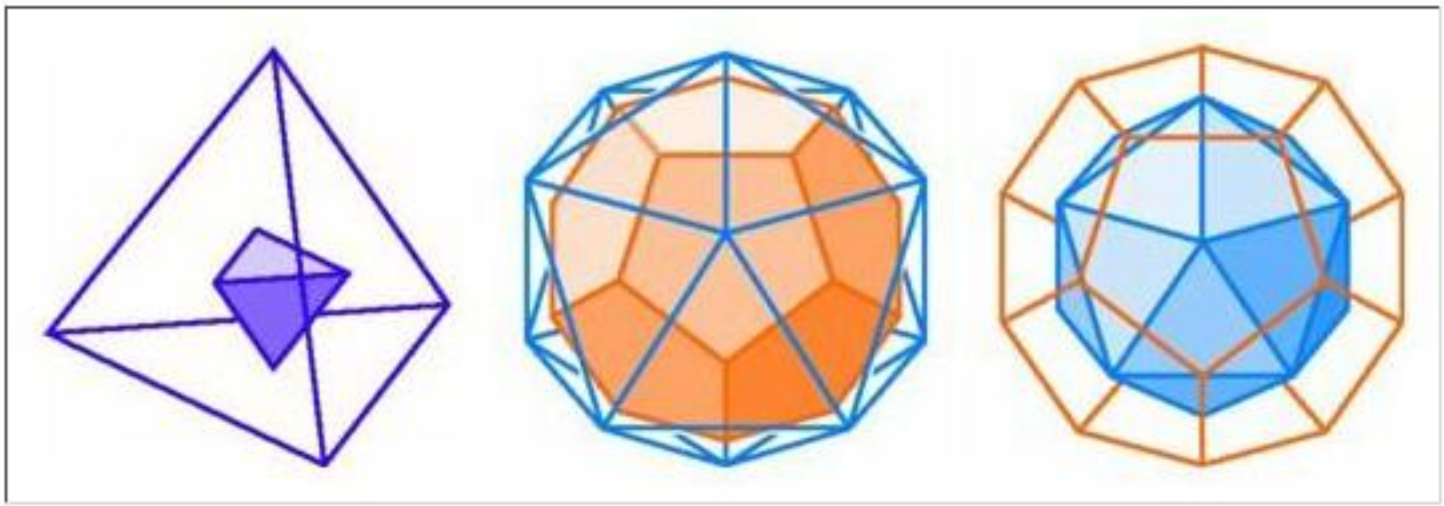
http://plus.maths.org/content/eulers-polyhedron-formula

Polyhedral duals

Polyhedra are associated into pairs called ***duals***, where the vertices of one correspond to the faces of the other. Starting with any given polyhedron, the dual of its dual is the original polyhedron.



The octahedron is dual to the cube (left); the cube is dual to the octahedron (right).

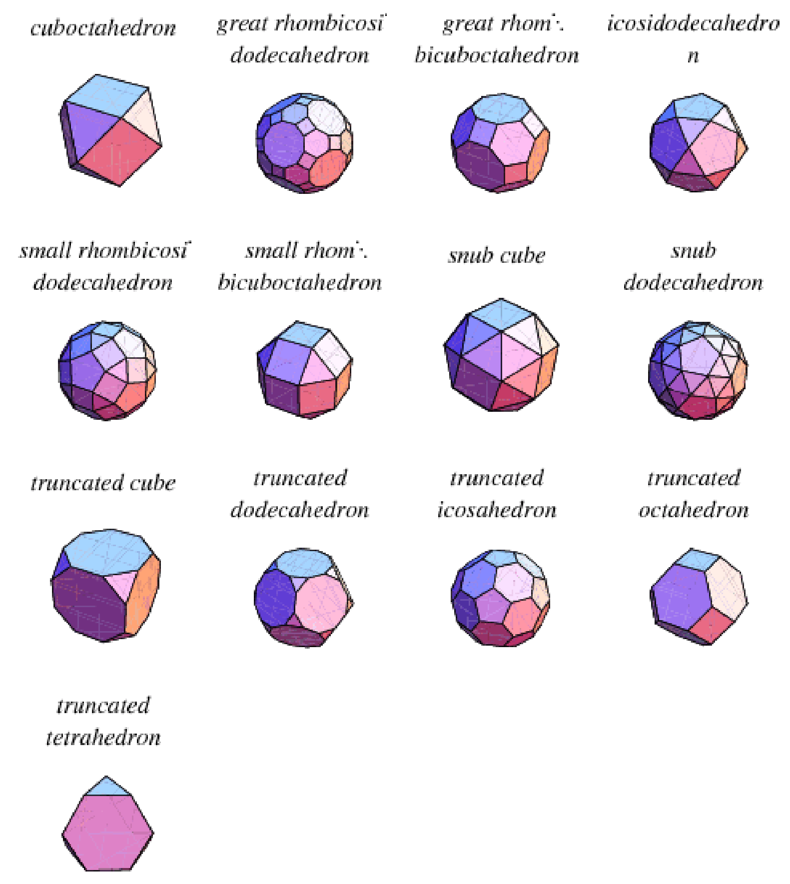


The self-dual tetrahedron (left); the dodecahedron is dual to the icosahedron (center); the icosahedron is dual to the dodecahedron (right).

http://www.math.brown.edu/~banchoff/Beyond3d/chapter5/section03.html

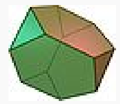
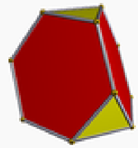
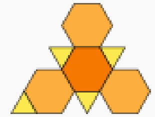
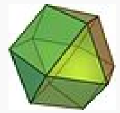
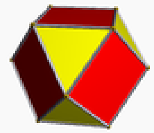
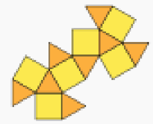
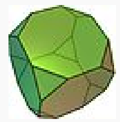
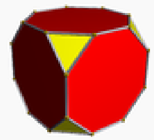
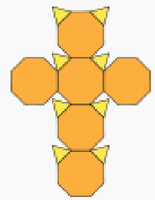
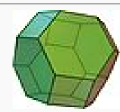
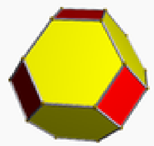
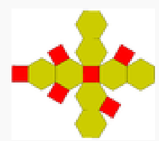
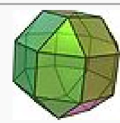
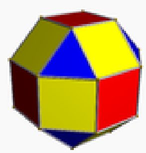
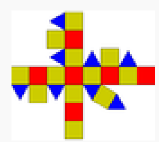
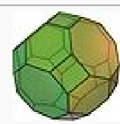
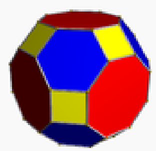
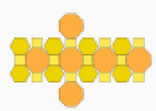
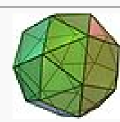
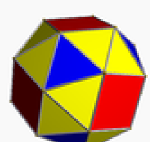
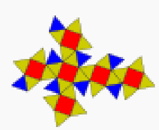
Archimedean solids

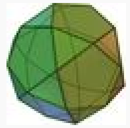
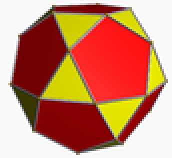
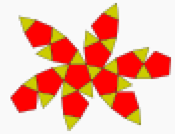
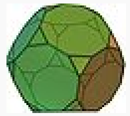
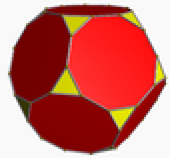
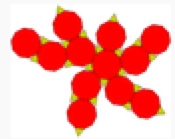
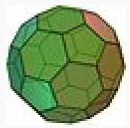
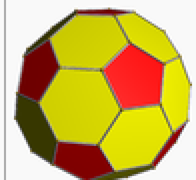
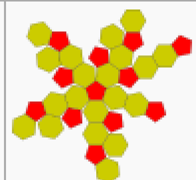
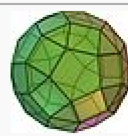
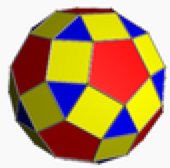
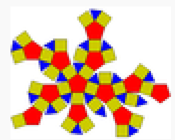
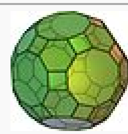
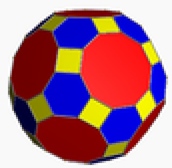
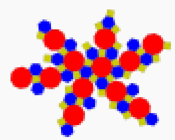
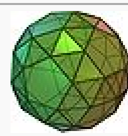
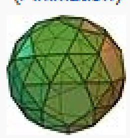
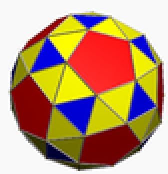
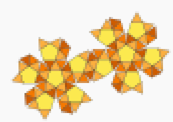
The Archimedean solids take their name from Archimedes, who discussed them in a now-lost work referred to by Pappus.



http://en.wikipedia.org/wiki/Archimedean_solid

Paper models: http://www.korthalsaltes.com/

Name (Vertex configuration)	Transparent	Solid	Net	Faces		Edges	Vertices	Point group
truncated tetrahedron (3.6.6)	 (Animation)			8	4 triangles 4 hexagons	18	12	T_d
cuboctahedron (3.4.3.4)	 (Animation)			14	8 triangles 6 squares	24	12	O_h
truncated cube or truncated hexahedron (3.8.8)	 (Animation)			14	8 triangles 6 octagons	36	24	O_h
truncated octahedron (4.6.6)	 (Animation)			14	6 squares 8 hexagons	36	24	O_h
rhombicuboctahedron or small rhombicuboctahedron (3.4.4.4)	 (Animation)			26	8 triangles 18 squares	48	24	O_h
truncated cuboctahedron or great rhombicuboctahedron (4.6.8)	 (Animation)			26	12 squares 8 hexagons 6 octagons	72	48	O_h
snub cube or snub hexahedron or snub cuboctahedron (2 chiral forms) (3.3.3.3.4)	 (Animation)  (Animation)			38	32 triangles 6 squares	60	24	O

icosidodecahedron (3.5.3.5)	 (Animation)			32	20 triangles 12 pentagons	60	30	I_h
truncated dodecahedron (3.10.10)	 (Animation)			32	20 triangles 12 decagons	90	60	I_h
truncated icosahedron (5.6.6)	 (Animation)			32	12 pentagons 20 hexagons	90	60	I_h
rhombicosidodecahedron or small rhombicosidodecahedron (3.4.5.4)	 (Animation)			62	20 triangles 30 squares 12 pentagons	120	60	I_h
truncated icosidodecahedron or great rhombicosidodecahedron (4.6.10)	 (Animation)			62	30 squares 20 hexagons 12 decagons	180	120	I_h
snub dodecahedron or snub icosidodecahedron (2 chiral forms) (3.3.3.3.5)	 (Animation)  (Animation)			92	80 triangles 12 pentagons	150	60	I

http://en.wikipedia.org/wiki/Archimedean_solid

Archimedean solids obtained by truncating Platonic solids

Truncation means cutting off the corners of a solid. We cut off identical lengths along each edge emerging from a vertex. This process adds a new face to the polyhedron. Each of the following pages explains the process in more detail.

Truncated Tetrahedron
Truncated Cube
Truncated Octahedron
Truncated Icosahedron
Truncated Dodecahedron

"Truncation all the way" - Rectification

Cuboctahedron
Icosidodecahedron

Archimedean solids obtained by truncating other Archimedean Solids

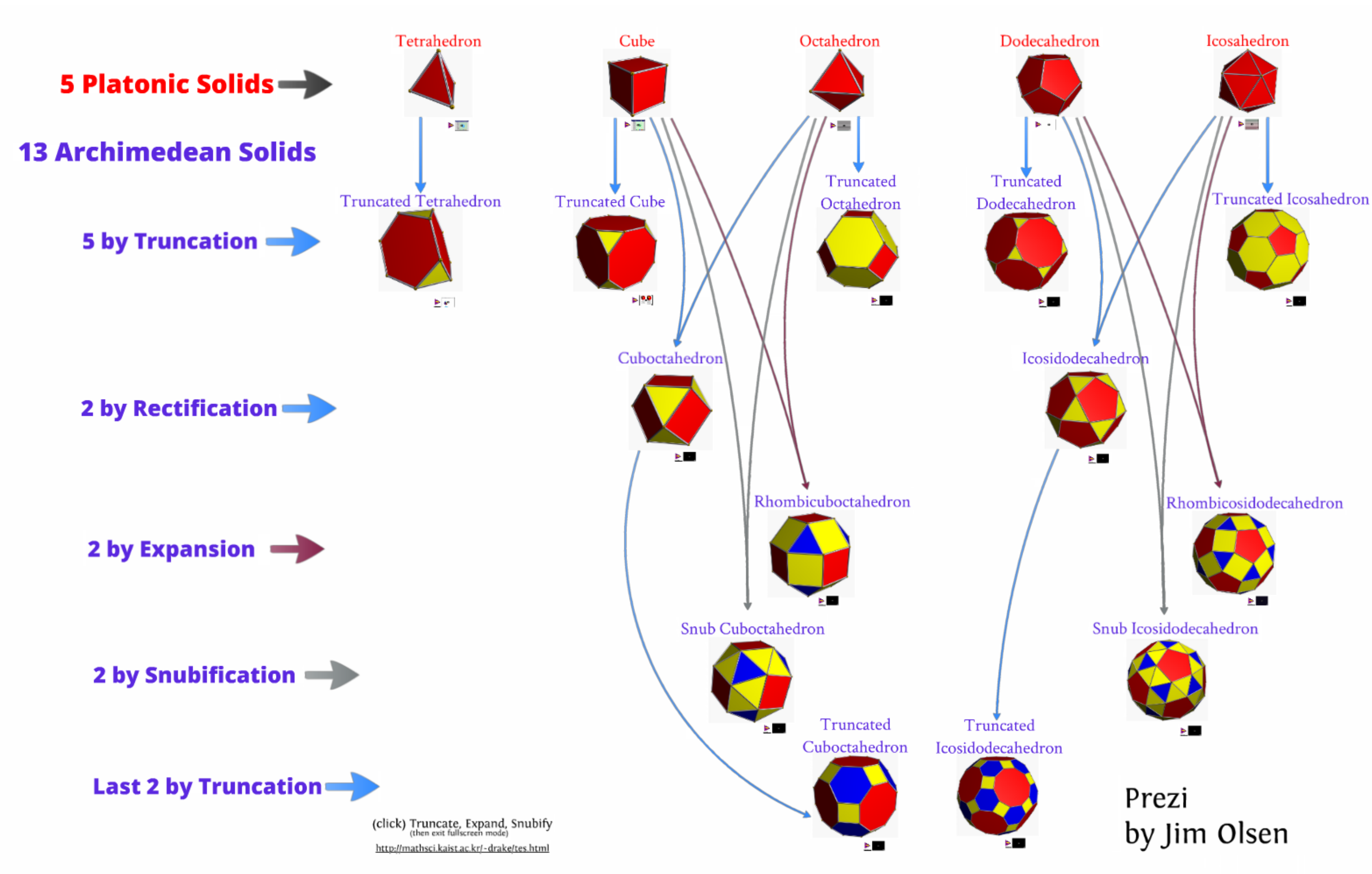
If we truncate the cuboctahedron or the icosidodecahedron, we will obtain four more solids.

Rhombicuboctahedron	*Expansion*
Rhombicosidodecahedron	*Expansion*
Rhombitruncated Cuboctahedron	or Truncated Cuboctahedron
Rhombitruncated Icosidodecahedron	or Truncated Icosidodecahedron

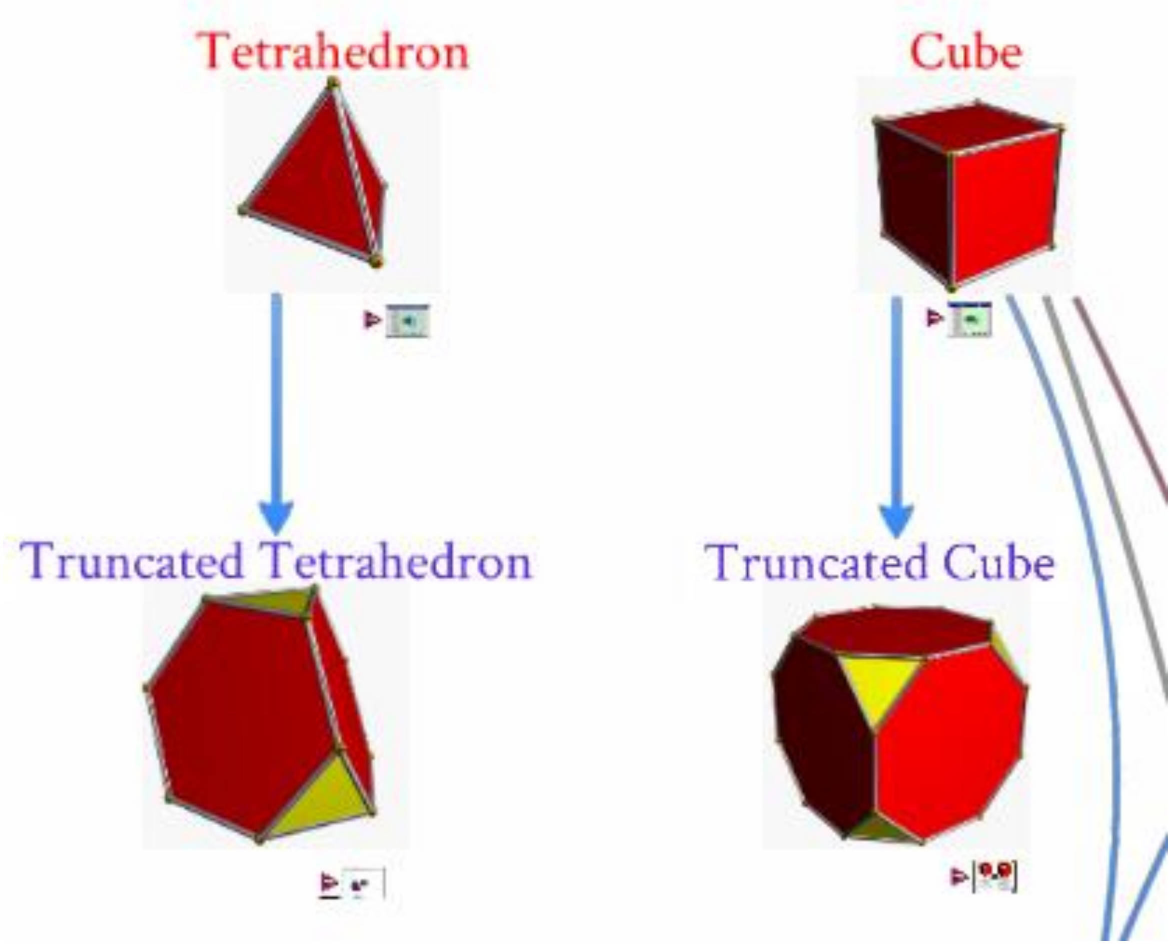
Archimedean solids obtained by "snubbing" Platonic Solids

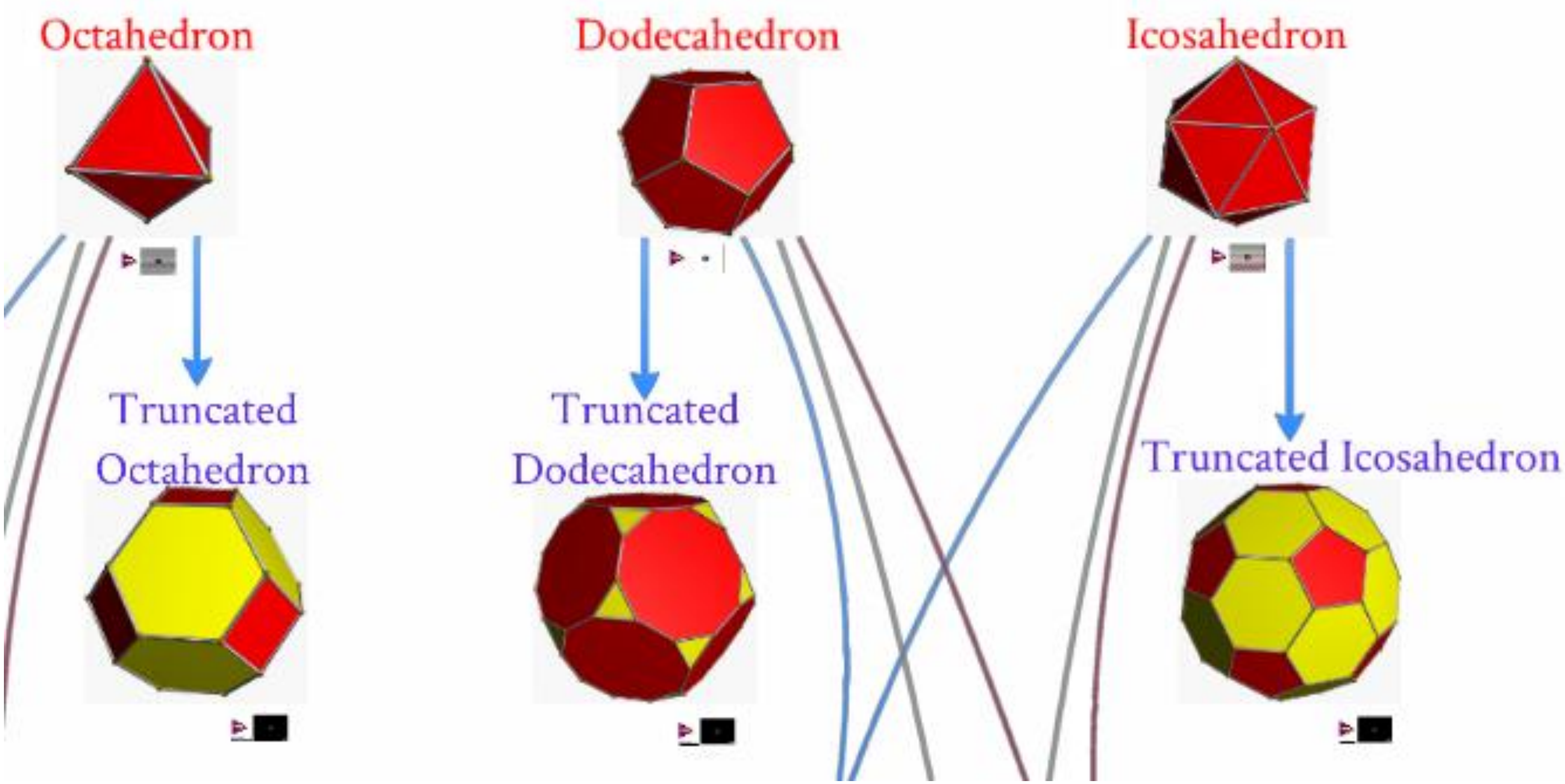
Snub Cube
Snub Dodecahedron

Platonic and Archimedean Solids: interesting properties

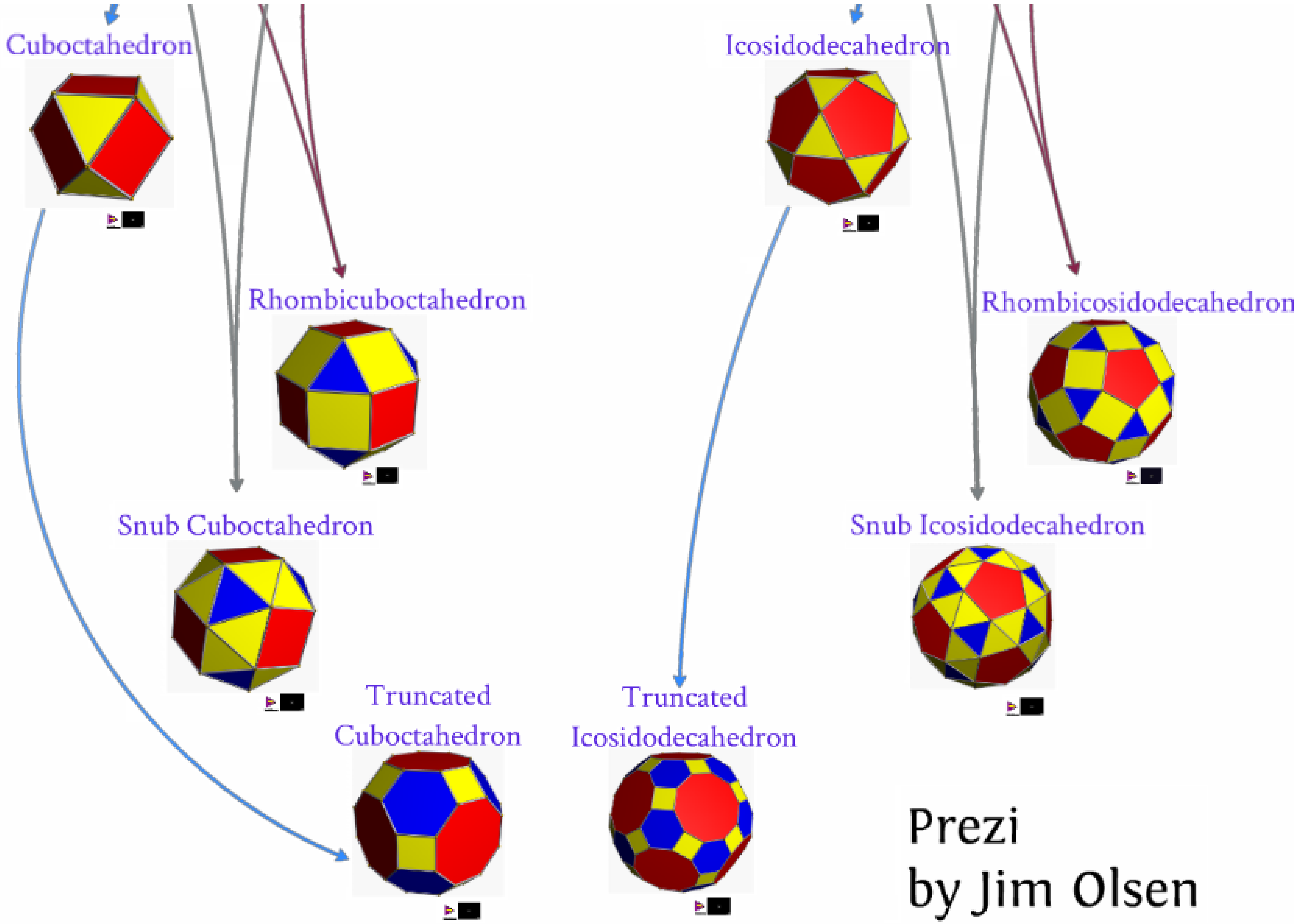


http://prezi.com/ur8uoh3azcem/platonic-and-archimedean-solids-interesting-properties/


Tetrahedron
Cube
Truncated Tetrahedron
Truncated Cube



Octahedron
Dodecahedron
Icosahedron
Truncated Octahedron
Truncated Dodecahedron
Truncated Icosahedron



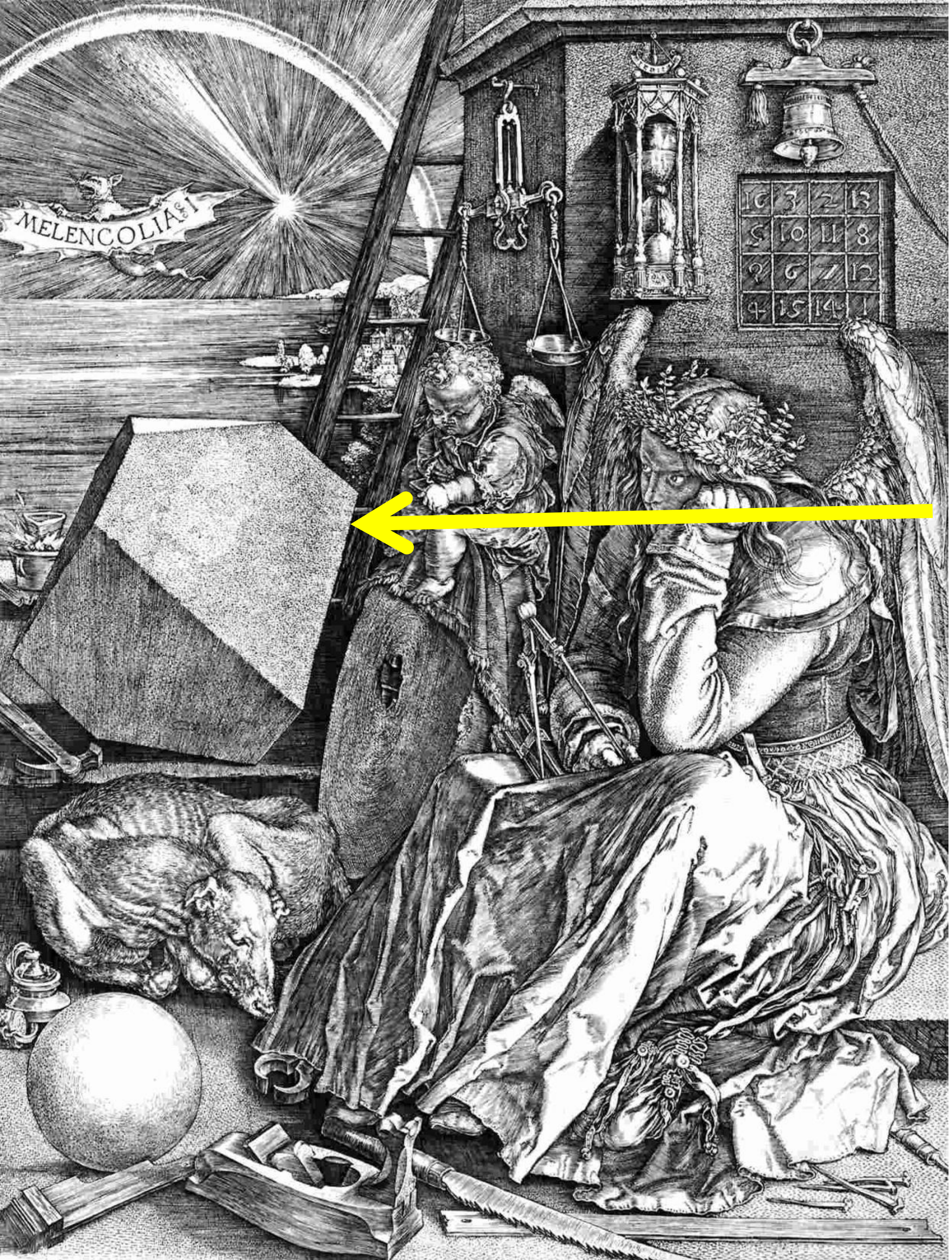
Prezi
by Jim Olsen

Archimedean solids by Truncation

http://www.screencast.com/users/DrO314/folders/Archimedean%20Solids
Truncate, Expand, Snubify - http://mathsci.kaist.ac.kr/~drake/tes.html

Archimedean solids by Expansion

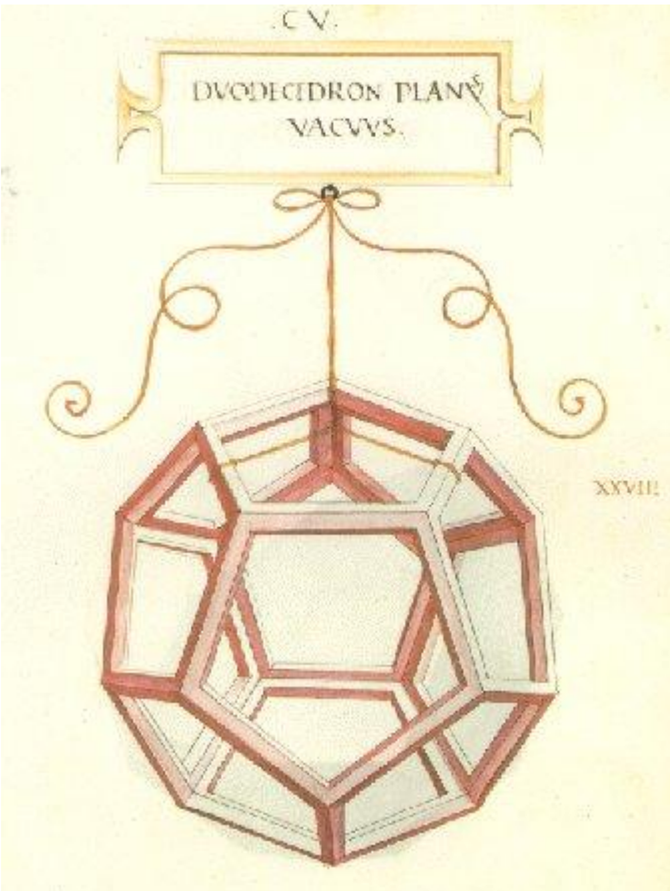
Archimedean solids by Snubification

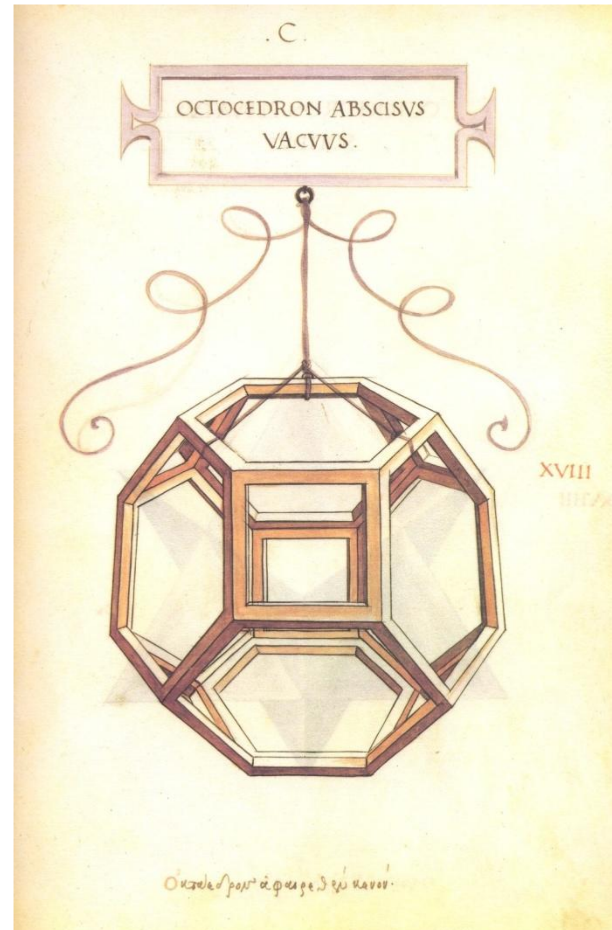


Melencolia I by the German Renaissance master Albrecht Dürer

The truncated rhombohedron with a faint human skull on it. This shape is now known as Dürer's solid; over the years, there have been numerous articles disputing the precise shape of this polyhedron

Da Vinci’s Polyhedra



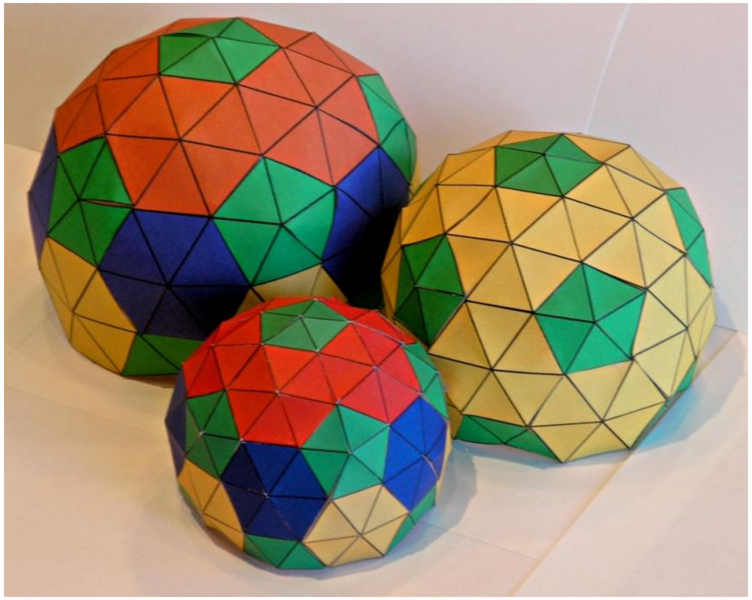


illustrations from Luca Pacioli’s 1509 book *The Divine Proportion*.

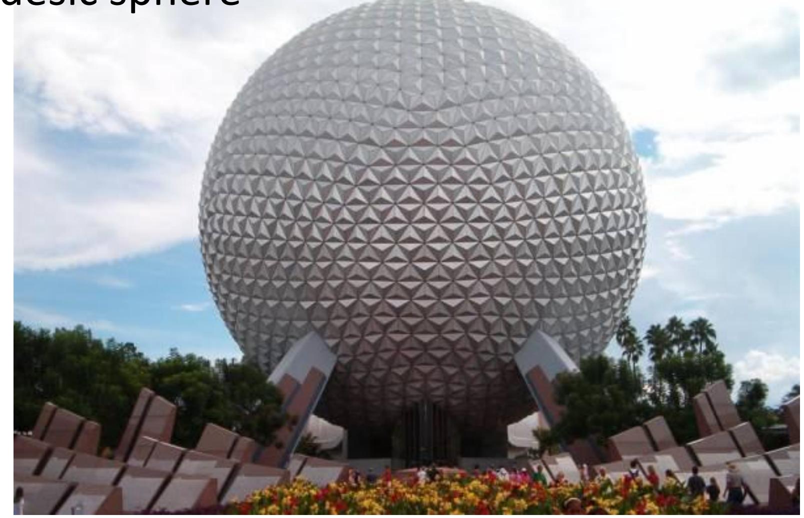
http://www.georgehart.com/virtual-polyhedra/leonardo.html

Geodesic dome

a geodesic dome design begins with an icosahedron inscribed in a hypothetical sphere, tiling each triangular face with smaller triangles, then projecting the vertices of each tile to the sphere.



Spaceship Earth at Epcot, Walt Disney World, a geodesic sphere

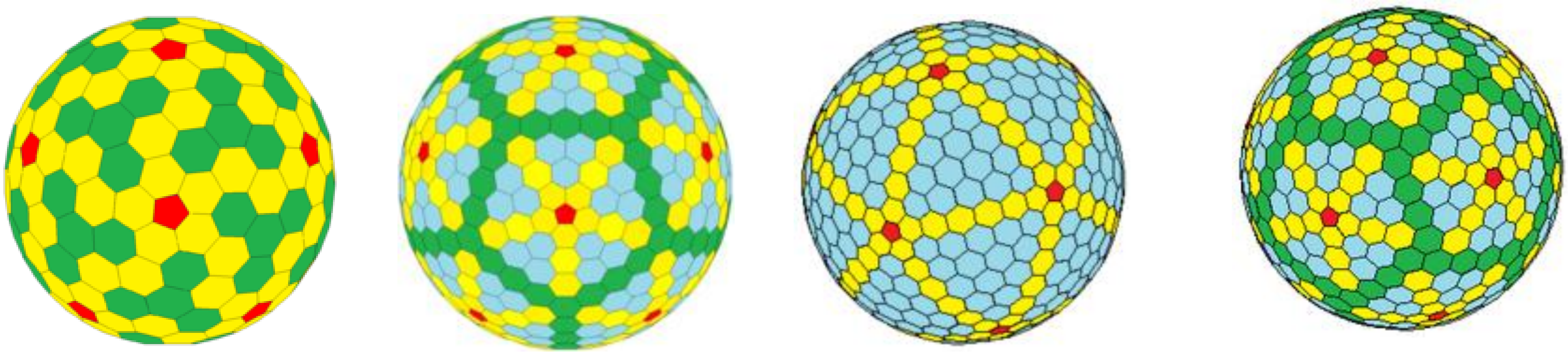


Paper domes: http://sci-toys.com/scitoys/scitoys/mathematics/dome/dome.html

Goldberg polyhedron

Goldberg polyhedron is a convex polyhedron made from hexagons and pentagons. They were first described by Michael Goldberg (1902–1990) in 1937. They are defined by three properties: each face is either a pentagon or hexagon, exactly three faces meet at each vertex, they have rotational icosahedral symmetry.

Icosahedral symmetry ensures that the pentagons are always regular, although many of the hexagons may not be. Typically all of the vertices lie on a sphere.



http://en.wikipedia.org/wiki/Goldberg_polyhedron

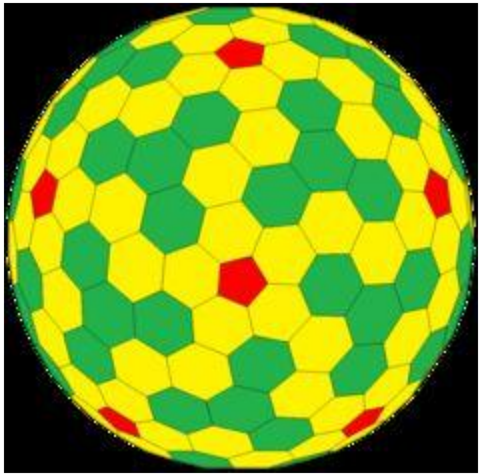
After 400 years, mathematicians find a new class of solid shapes

The new discovery comes from researchers who were inspired by finding such interesting polyhedra in their own work that involved the human eye.

......

During this work, Schein came across the work of 20th century mathematician Michael Goldberg who described a set of new shapes, which have been named after him, as Goldberg polyhedra. The easiest Goldberg polyhedron to imagine looks like a blown-up football, as the shape is made of many pentagons and hexagons connected to each other in a symmetrical manner.

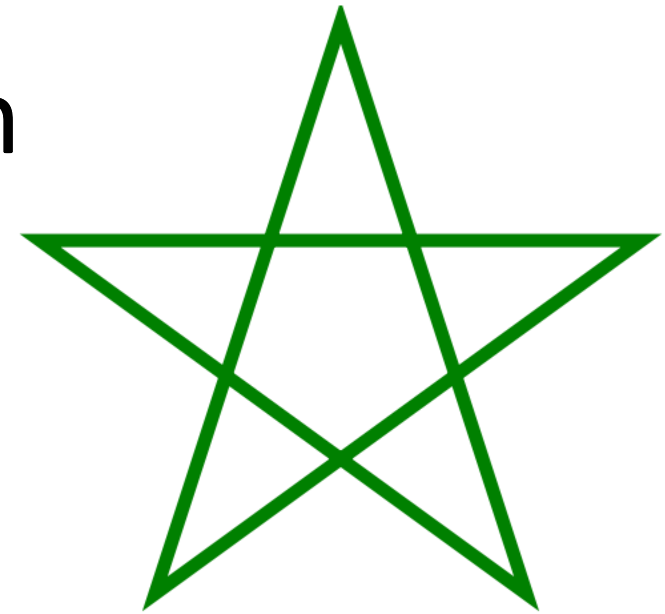
.....in a new paper in the Proceedings of the National Academy of Sciences, Schein and his colleague James Gayed have described that a fourth class of convex polyhedra, which given Goldberg's influence they want to call Goldberg polyhedra, even at the cost of confusing others.



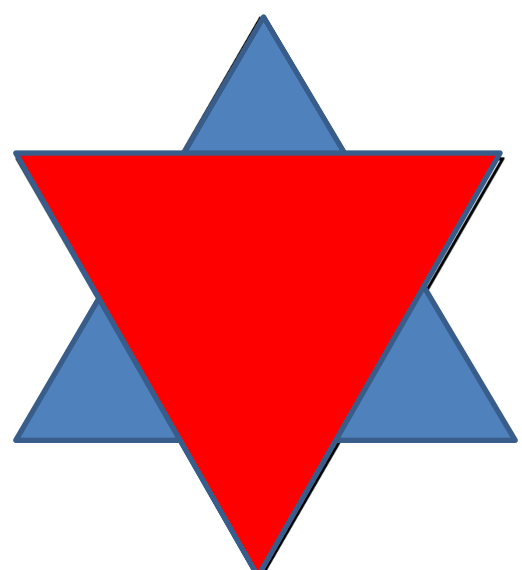
http://theconversation.com/after-400-years-mathematicians-find-a-new-class-of-solid-shapes-23217

Stellation

- In 1619 Kepler defined stellation for polygons and polyhedra, as the process of extending edges or faces until they meet to form a new polygon or polyhedron.



The pentagram, {5/2}, is the only stellation of a pentagon

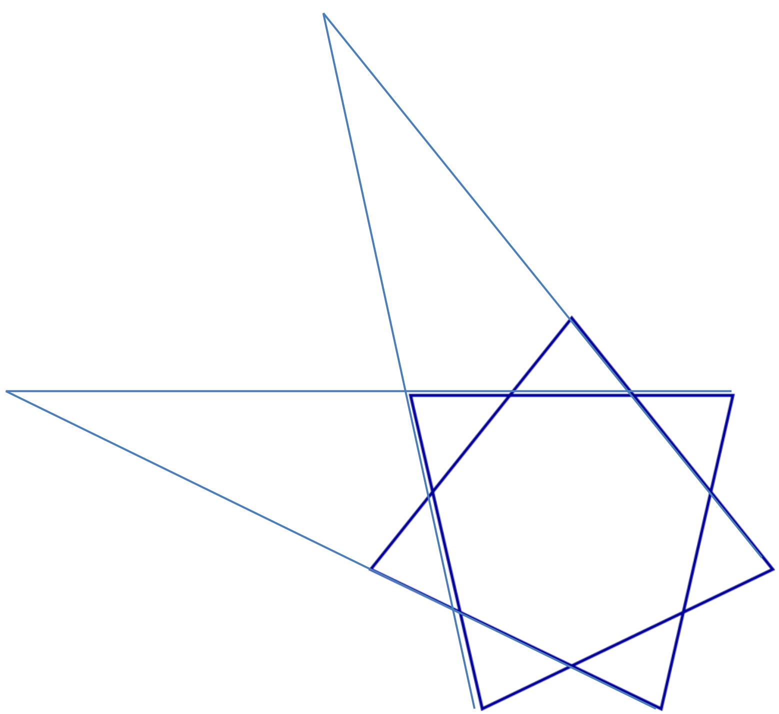


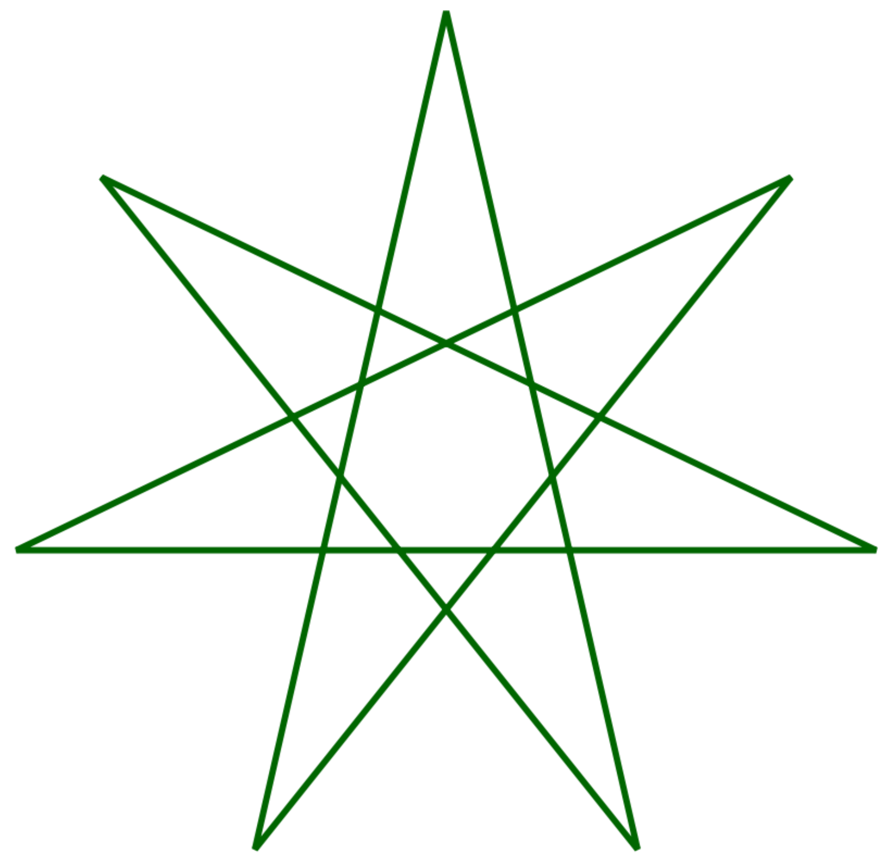
The hexagram, {6/2}, the stellation of a hexagon and a compound of two triangles.

Stellation – an example of multiple forms

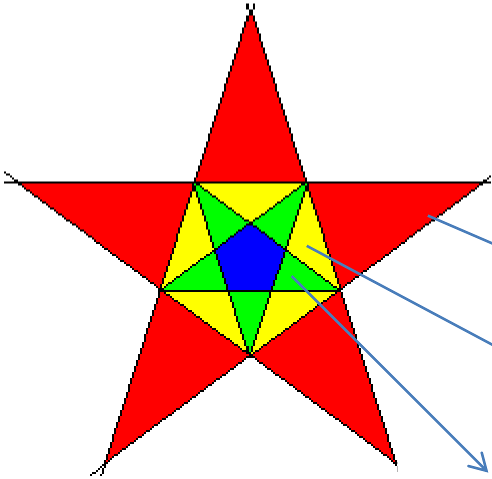
The heptagon has two heptagrammic forms:

{7/2}, {7/3}

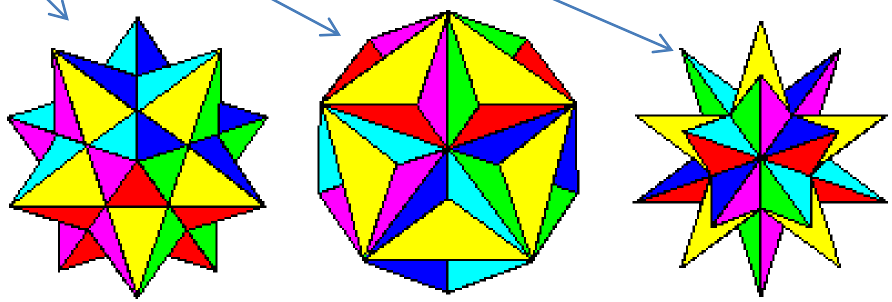




Stellating Polyhedra



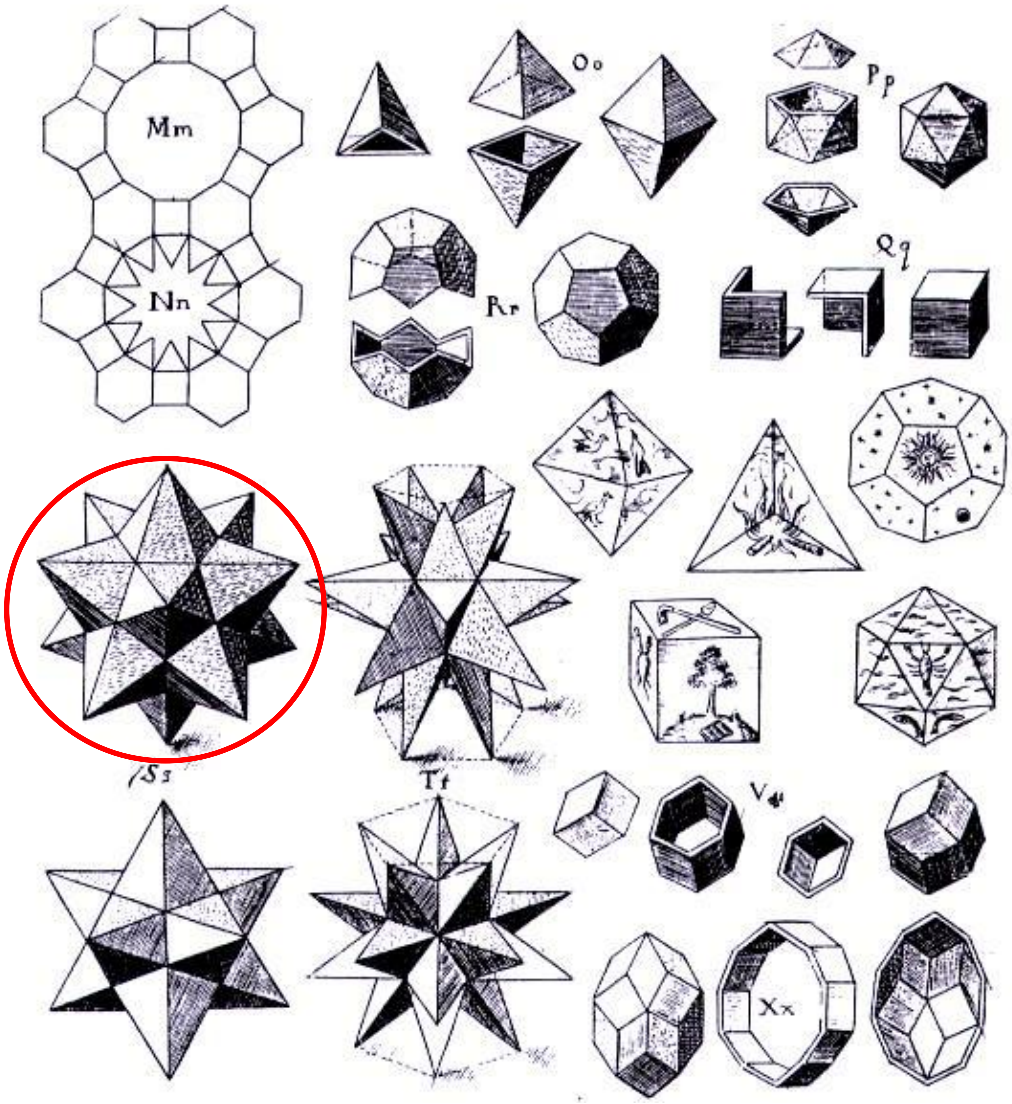
Here we see a dodecahedron face (blue) with the intersections of all other faces indicated. This is a common way to show the possible stellations of a solid. We see that there are three distinct groups of cells.

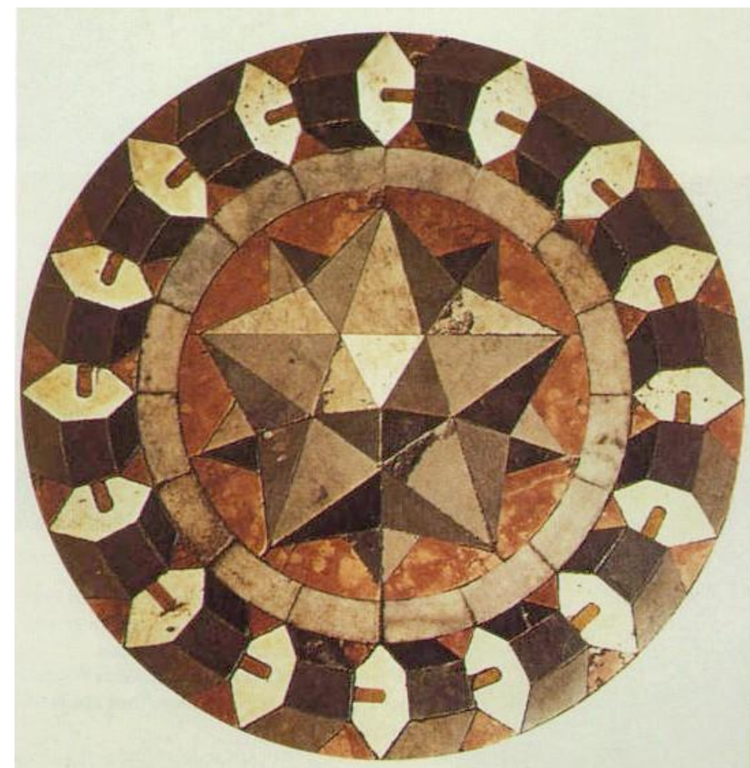


The three stellations of the dodecahedron are shown here. The first or innermost is the *small stellated dodecahedron*, discovered by Kepler. Next is the *great dodecahedron*, discovered by Poinsot in 1809. This is obtained by continuing the star planes of the small stellated dodecahedron outward until they meet to form the next set of pentagons. If we extend these pentagons, we get the stellation on the right, the *great stellated dodecahedron*, also discovered by Kepler.

https://www.uwgb.edu/dutchs/symmetry/stellate.htm illustration, from Kepler's 1619 book, *Harmonice Mundi*, also graphically shows the Platonic associations of the regular solids with the classical elements

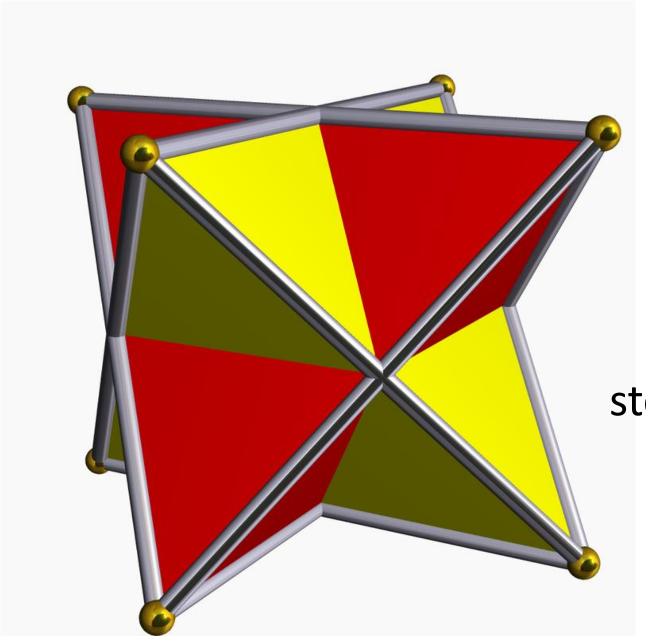
http://www.georgehart.com/virtual-polyhedra/kepler.html



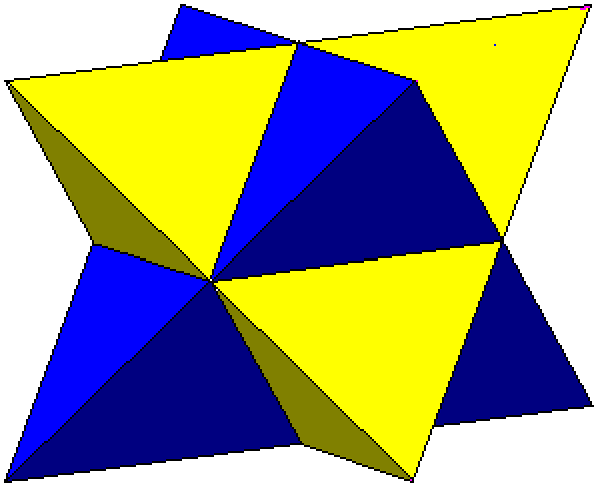


Marble floor mosaic
Basilica of St Mark Venice

Kepler also stellated the regular octahedron to obtain the stella octangula, a regular compound of two tetrahedra.

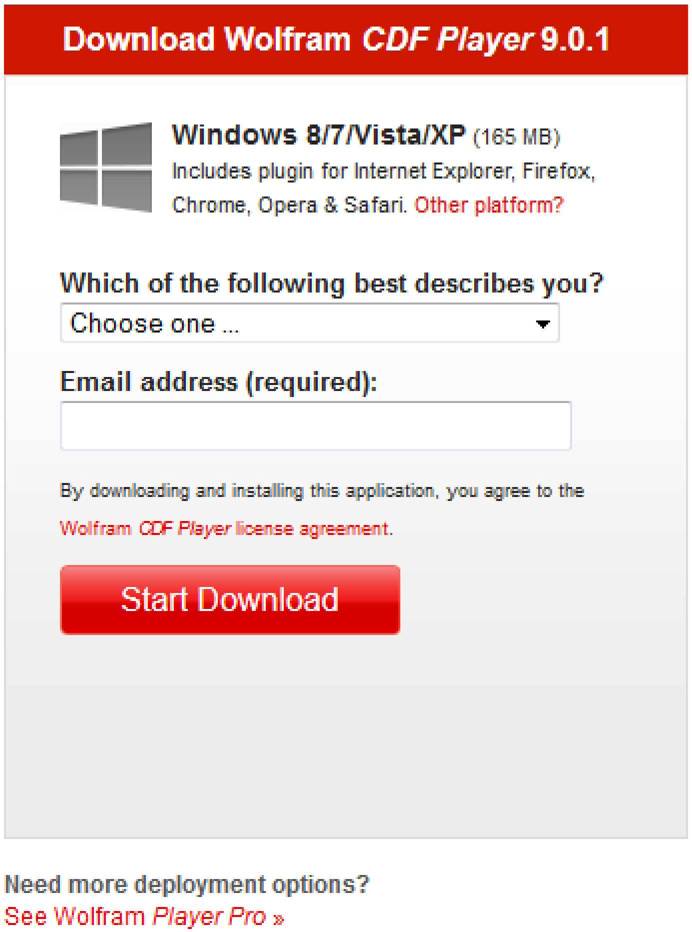


stella octangula



In this version it is easier to see the intersecting tetrahedrons

Wolfram CDF Player™



Visualisation software:

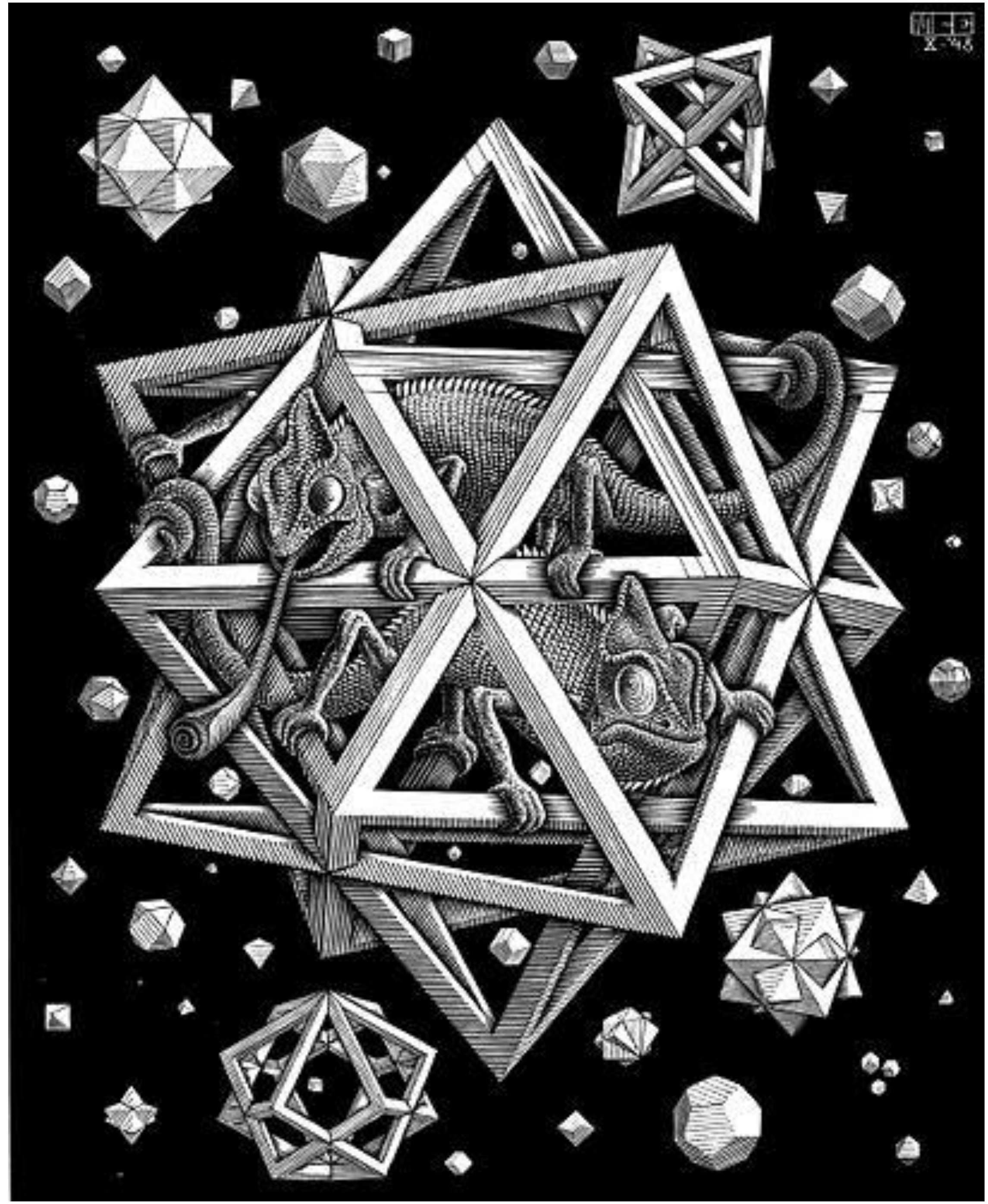
Wolfram Computable Document Format:

http://www.wolfram.com/cdf/

Visualisation software from Wolfram

Escher again!

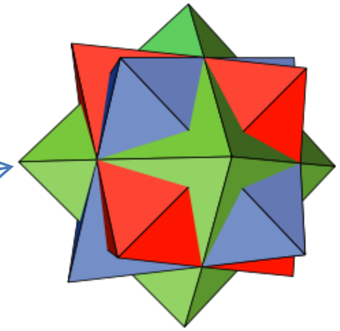
Name that Polyhedron!

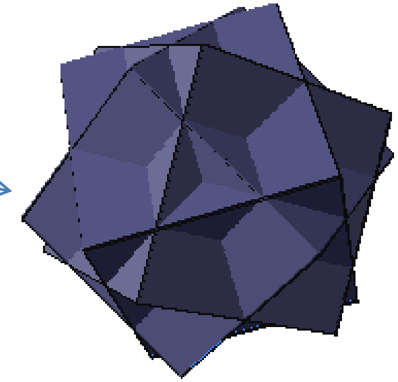


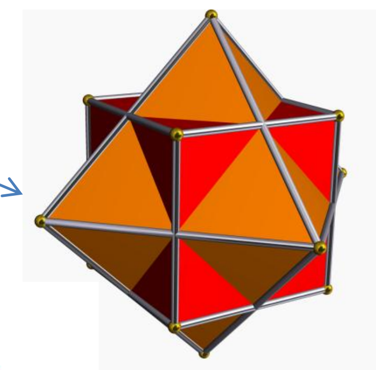
http://www.georgehart.com/virtual-polyhedra/escher.html

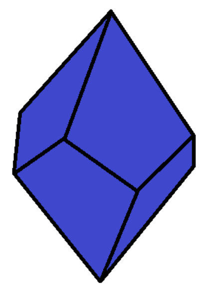
Answers:

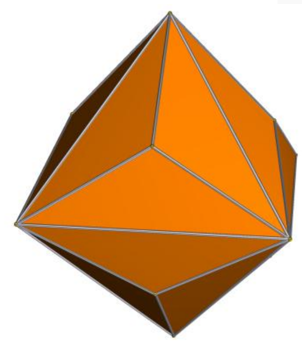
- compound of three octahedra,
(both a solid and an edge model)
- compound of two cubes with common 3-fold axis,
(both a solid and an edge model)
- stella octangula (compound of two tetrahedra),
(both a solid and an edge model)
- compound of cube and octahedron,
- rhombic dodecahedron,
- cuboctahedron,
- rhombicuboctahedron,
- square trapezohedron,
- trapezoidal icositetrahedron,
- triakis octahedron,
- all five platonic solids.











http://www.georgehart.com/virtual-polyhedra/escher.html

Making Origami Platonic and Archimedean solids

http://www.origami-resource-center.com/origami-polyhedra-design.html

(examples and review of book by John Montroll)

Video here for stellated octahedron http://www.youtube.com/watch?v=6LVX6zWDcJk

A full range appears at http://www.mathigon.org/origami/

Not all construction links shown work!

Making an origami octahedron

http://www.youtube.com/watch?v=phhVl-N9M4Y

Stella's Polyhedra Bookcase
http://www.software3d.com/Bookcase.php

If you want to try out Wolfram's CDF software you can get it from here
http://www.wolfram.com/cdf/

Once loaded there are examples here:
http://demonstrations.wolfram.com/index.html

Next time.....

- Probability and how not to gamble
  - What are the odds and why you should only bet on cetainties
- Special numbers and why they are special
  - Pi, the Golden Ratio and others
- Paradoxically thinking
  - Logic and paradoxes
- Games to play and how to win them
  - Games of strategy
- To infinity and beyond
  - Different kinds of infinity! Mandelbrot